

IDEMPOTENT STATES ON LOCALLY COMPACT QUANTUM GROUPS

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ABSTRACT. Correspondence between idempotent states and expected right-invariant subalgebras is extended to non-coamenable, non-unimodular locally compact quantum groups; in particular left convolution operators are shown to automatically preserve the right Haar weight.

Idempotent states on finite and compact quantum groups, generalising the idempotent probability measures on compact groups, have been studied in a series of papers [FS_{1–2}] and [FST] (see also the survey [Sal₁], where one can find the probabilistic and harmonic-analytic motivations behind investigating such objects). In the article [SaS] the main results related to idempotent states were extended to the *locally compact* case, under the assumption that the locally compact quantum group \mathbb{G} in question is *unimodular* and *coamenable*. The first of these properties is automatically satisfied in the compact case, whereas the second, indeed assumed in [FST] and [FS₂], means that all idempotent states may be viewed as bounded functionals on the C^* -algebra $C_0(\mathbb{G})$, and not, as is the case in general, on its universal counterpart, $C_0^u(\mathbb{G})$.

In this short paper we show that in fact the main results of [SaS] hold also when both of these assumptions are dropped. We also connect the von Neumann algebraic picture, i.e. working with the algebra $L^\infty(\mathbb{G})$ of ‘essentially bounded functions’ on \mathbb{G} , with the C^* -algebraic one. Thus we prove the following theorem (the detailed explanation of the terms used below may be found in the main body of the paper; note that expected C^* -subalgebras are automatically non-zero).

Theorem 1. *Let \mathbb{G} be an arbitrary locally compact quantum group. There is a one-to-one correspondence between the following objects:*

- (i) *idempotent states in $\text{Prob}^u(\mathbb{G})$;*
- (ii) *right-invariant expected C^* -subalgebras of $C_0(\mathbb{G})$;*
- (iii) *right-invariant expected von Neumann subalgebras of $L^\infty(\mathbb{G})$;*
- (iv) *left-invariant expected C^* -subalgebras of $C_0(\mathbb{G})$;*
- (v) *left-invariant expected von Neumann subalgebras of $L^\infty(\mathbb{G})$.*

The key part in the proof of the above result is based on the following theorem.

Theorem 2. *Let $\omega \in \text{Prob}^u(\mathbb{G})$ be an idempotent state. Then the associated left multiplier L_ω preserves both the left and the right Haar weight.*

Note one immediate corollary, which admits an obvious left-sided counterpart.

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Corollary 3. *A right-invariant ψ -expected C^* -subalgebra of $C_0(\mathbb{G})$ is automatically expected. Similarly, a right-invariant ψ -expected von Neumann subalgebra of $L^\infty(\mathbb{G})$ is automatically expected.*

We also deduce from the main theorem some properties enjoyed by idempotent states.

Proposition 4. *Idempotent states in $\text{Prob}^u(\mathbb{G})$ are invariant with respect to the antipode S^u , the unitary antipode R^u and the scaling group $\{\tau_t^u : t \in \mathbb{R}\}$.*

Finally we obtain the following characterisation of Haar idempotent states, i.e. those that arise as Haar states on compact quantum subgroups of \mathbb{G} .

Theorem 5. *Let $\omega \in \text{Prob}^u(\mathbb{G})$ be an idempotent state. The following conditions are equivalent:*

- (i) ω is a Haar idempotent;
- (ii) the null space $N_\omega = \{a \in C_0^u(\mathbb{G}) : \omega(a^*a) = 0\}$ is a two-sided ideal;
- (iii) the right-invariant expected C^* -subalgebra of $C_0(\mathbb{G})$ associated to ω is symmetric.

Here a remark is in place: after the first version of this article was circulated, P. Kasprzak and F. Khosravi shared with us a draft of their paper [KaK]. The correspondence (i) \iff (iii) is essentially contained in that paper. Note also that Section 4 of [KaK] improves Corollary 3.

The fact that the coamenability assumption can be dropped in the compact case was also observed in [FLS] – there it is much simpler, as one can use essentially purely algebraic methods. The locally compact non-coamenable context necessitates applying the theory of normal (left) multipliers on $L^\infty(\mathbb{G})$, as developed in [JNR] and [Daw₁]. Finally we note that as some of the proofs in the non-coamenable setting use the techniques and arguments developed in [SaS], in the text below we often simply refer to the appropriate parts of that paper.

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1. NOTATION, TERMINOLOGY AND BACKGROUND

Throughout the paper \mathbb{G} will be a locally compact quantum group in the sense of Kustermans and Vaes, described via the ‘algebra of essentially bounded functions’ $L^\infty(\mathbb{G})$ and the ‘algebra of continuous functions vanishing at infinity’ $C_0(\mathbb{G}) \subset L^\infty(\mathbb{G})$, where $C_0(\mathbb{G})$ is a (usually non-unital) C^* -algebra and $L^\infty(\mathbb{G})$ a von Neumann algebra equipped with a *co-product* $\Delta : L^\infty(\mathbb{G}) \rightarrow L^\infty(\mathbb{G}) \bar{\otimes} L^\infty(\mathbb{G})$. We will in general use the notation and terminology of [DFSW] and refer to the precise definitions of all the objects appearing below to [KuV], [DFSW] and [SaS]. The *left* and the *right Haar weights* of \mathbb{G} will be denoted by symbols ϕ and ψ , respectively. The von Neumann algebra $L^\infty(\mathbb{G})$, so also $C_0(\mathbb{G})$ and the multiplier algebra $C_b(\mathbb{G}) := M(C_0(\mathbb{G}))$, act on the GNS Hilbert space $L^2(\mathbb{G})$ of ϕ . We will use the standard notation $\mathcal{N}_\phi = \{x \in L^\infty(\mathbb{G}) : \phi(x^*x) < \infty\}$. The universal version of $C_0(\mathbb{G})$ will be written as $C_0^u(\mathbb{G})$, the corresponding coproduct as Δ_u , and the *reducing morphism* from $C_0^u(\mathbb{G})$ onto $C_0(\mathbb{G})$ as Λ . The dual space of $C_0^u(\mathbb{G})$ will be denoted by $M^u(\mathbb{G})$, and the corresponding state space by $\text{Prob}^u(\mathbb{G})$ – recall that $M^u(\mathbb{G})$ is equipped with a natural Banach algebra structure given by the convolution product $\mu \star \nu := (\mu \otimes \nu) \circ \Delta_u$ for $\mu, \nu \in M^u(\mathbb{G})$. An

element $\omega \in \text{Prob}^u(\mathbb{G})$ is called an *idempotent state* if $\omega \star \omega = \omega$. Also the predual of $L^\infty(\mathbb{G})$, denoted by $L^1(\mathbb{G})$, admits a similar convolution product; we will need at some point the fact that $L^1(\mathbb{G})$ admits a dense subalgebra $L^1_\#(\mathbb{G})$, which allows a natural involution, related to the *antipode* S of $L^\infty(\mathbb{G})$. The *unitary antipode* of $L^\infty(\mathbb{G})$ will be denoted by R , the *modular automorphisms of the left Haar weight* by σ_t ($t \in \mathbb{R}$), the *scaling automorphisms* by τ_t ($t \in \mathbb{R}$) and the *modular element* of \mathbb{G} by δ ; all these have the counterparts on the universal level, denoted respectively by S^u , R^u , σ_t^u , τ_t^u and δ_u (see [Kus₂] for details).

Each locally compact quantum group admits a *dual locally compact quantum group* $\widehat{\mathbb{G}}$. The *multiplicative unitary* belonging to the multiplier algebra $M(C_0(\mathbb{G}) \otimes C_0(\widehat{\mathbb{G}}))$ implementing the coproduct will be denoted by W ; we will also use its semi-universal version $W \in M(C_0^u(\mathbb{G}) \otimes C_0(\widehat{\mathbb{G}}))$ and the universal version $W \in M(C_0^u(\mathbb{G}) \otimes C_0^u(\widehat{\mathbb{G}}))$.

We say that a locally compact quantum group \mathbb{H} is *compact* if the algebra $C_0(\mathbb{H})$ is unital. We then denote the respective reduced/universal C^* -algebras of ‘functions’ on \mathbb{H} by $C(\mathbb{H})$ and $C^u(\mathbb{H})$; each compact quantum group \mathbb{H} admits a *Haar state* $h_{\mathbb{H}} \in \text{Prob}^u(\mathbb{H})$, a unique state such that $h_{\mathbb{H}} \star \mu = \mu \star h_{\mathbb{H}} = h_{\mathbb{H}}$ for all $\mu \in \text{Prob}^u(\mathbb{H})$. A compact quantum group \mathbb{G} is said to be a (*closed*) *quantum subgroup* of a locally compact quantum group \mathbb{G} if there exists a surjective morphism $\pi : C_0^u(\mathbb{G}) \rightarrow C^u(\mathbb{H})$ intertwining the respective coproducts. In such a case it is easy to see that $\omega := h_{\mathbb{H}} \circ \pi$ is an idempotent state; the idempotent states which arise in this way are called *Haar idempotents*.

A C^* -subalgebra $\mathbb{C} \subset C_0(\mathbb{G})$ is said to be *right invariant* if $(\text{id} \otimes \mu)(\Delta(\mathbb{C})) \subset \mathbb{C}$ for all $\mu \in C_0(\mathbb{G})^*$. It is said to be *ϕ -expected* (respectively, *ψ -expected*) if there exists a conditional expectation E from $C_0(\mathbb{G})$ onto \mathbb{C} that is ϕ -preserving (respectively, ψ -preserving) and simply *expected* if there exists a conditional expectation E onto \mathbb{C} that is both ψ -preserving and ϕ -preserving; note that expected subalgebras are necessarily non-trivial. Similarly a von Neumann subalgebra $\mathbb{D} \subset L^\infty(\mathbb{G})$ is said to be *right invariant* if $(\text{id} \otimes \mu)(\Delta(\mathbb{D})) \subset \mathbb{D}$ for all $\mu \in L^1(\mathbb{G})$ (equivalently, $\Delta(\mathbb{D}) \subset \mathbb{D} \overline{\otimes} L^\infty(\mathbb{G})$) and we call it *ϕ -expected* (or *ψ -expected*, or *expected*) if the respective expectations from $L^\infty(\mathbb{G})$ onto \mathbb{D} are in addition normal. For a discussion of weight preservation etc., we refer to [SaS]. Finally we say that a right-invariant C^* -subalgebra $\mathbb{C} \subset C_0(\mathbb{G})$ is *symmetric* if the following holds:

$$W^*(\mathbb{C} \otimes 1)W \subset M(\mathbb{C} \otimes C_0(\widehat{\mathbb{G}})).$$

Note that Proposition 3.1 of [KaK] (or rather its right version) shows that non-zero right-invariant subalgebras of $C_0(\mathbb{G})$ are automatically nondegenerate.

We now summarise the main facts concerning the unital, completely positive, normal left multipliers on $L^\infty(\mathbb{G})$, proved in papers [JNR] and [Daw₁] and gathered in [DFSW].

Theorem 6. *Let $L : L^\infty(\mathbb{G}) \rightarrow L^\infty(\mathbb{G})$ be a normal unital completely positive map. Then the following are equivalent:*

- (i) $L = T^*$, where $T : L^1(\mathbb{G}) \rightarrow L^1(\mathbb{G})$ is a bounded left module map;
- (ii)

$$\Delta \circ L = (L \otimes \text{id}) \circ \Delta; \tag{1}$$

- (iii) *there exists $a \in C_b(\widehat{\mathbb{G}})$ such that $(L \otimes \text{id})(W) = (1 \otimes a)W$;*
- (iv) *there exists $\mu \in \text{Prob}^u(\mathbb{G})$ such that*

$$L(x) = (\mu \otimes \text{id})(W^*(1 \otimes x)W), \quad x \in L^\infty(\mathbb{G}). \tag{2}$$

In the above case we in fact have $a = (\mu \otimes \text{id})(W)$. Note further that the formula (2) defines a normal unital completely positive map on $L^\infty(\mathbb{G})$ for any $\mu \in \text{Prob}^u(\mathbb{G})$, to be denoted L_μ in what follows; the correspondence $\mu \mapsto L_\mu$ is injective.

We need some more properties of the maps described in the above theorem, which we will call *left multipliers* of $L^\infty(\mathbb{G})$.

Proposition 7. *Let $\mu \in \text{Prob}^u(\mathbb{G})$. The map L_μ defined above preserves the left Haar weight ϕ . Moreover, it restricts to a completely positive nondegenerate (in the sense of [SaS]) map on $C_0(\mathbb{G})$. If we consider the (completely positive, nondegenerate) map $L_\mu^u : C_0^u(\mathbb{G}) \rightarrow C_0^u(\mathbb{G})$ given by $L_\mu^u = (\mu \otimes \text{id}) \circ \Delta_u$, then we have*

$$\Lambda \circ L_\mu^u = L_\mu \circ \Lambda. \quad (3)$$

Finally if $\nu \in \text{Prob}^u(\mathbb{G})$ is another state, then

$$L_\mu \circ L_\nu = L_{\nu \star \mu}. \quad (4)$$

Proof. The first fact stated above is Lemma 3.4 of [KNR]. The second is noted in [DFSW] and also in [Daw₂]. The third is a consequence of the following formula, established in Proposition 6.2 of [Kus₂]:

$$(\text{id} \otimes \Lambda) \circ \Delta_u(x) = W^*(\text{id} \otimes \Lambda(x))W, \quad x \in C_0^u(\mathbb{G})$$

(note that the maps θ appearing in [Kus₂] disappear here as we tacitly assume that $C_0(\widehat{\mathbb{G}})$ is represented on $L^2(\mathbb{G})$). Finally the last relation is easily checked on the level of the ‘universal’ maps L_μ^u and then follows by the equality (3), normality of the multipliers and weak*-density of $C_0(\mathbb{G})$ in $L^\infty(\mathbb{G})$. \square

Given an element $\nu \in C_0(\mathbb{G})^*$ we will write simply L_ν for an operator formally defined as $L_{\nu \circ \Lambda}$; it is easy to see that then $L_\nu = (\nu \otimes \text{id}) \circ \Delta$ on $C_0(\mathbb{G})$ (also on $L^\infty(\mathbb{G})$ with a suitable interpretation of the right-hand side).

Finally note that we can of course consider the corresponding (unital completely positive) *right multipliers*, to be denoted R_μ for $\mu \in \text{Prob}^u(\mathbb{G})$. Then the left multipliers commute with the right ones, as can be seen for example from condition (i) in Theorem 6 and the fact that products of elements in $L^1(\mathbb{G})$ are dense in $L^1(\mathbb{G})$.

2. PROOFS OF THE MAIN RESULTS

An important step towards the main theorem is Theorem 2, which we will prove first. To this end, we need a few lemmas.

Recall that the modular elements δ and δ_u are unbounded, strictly positive operators affiliated with $C_0(\mathbb{G})$ and $C_0^u(\mathbb{G})$, respectively.

Lemma 8. *Let $\omega \in \text{Prob}^u(\mathbb{G})$ be an idempotent state. Then for every $t \in \mathbb{R}$ and $a \in C_0^u(\mathbb{G})$*

$$L_\omega^u(\delta_u^{it} a) = \delta_u^{it} L_\omega^u(a) \quad \text{and} \quad L_\omega^u(a \delta_u^{it}) = L_\omega^u(a) \delta_u^{it}.$$

Proof. We prove only the first identity, the second being similar. As ω is an idempotent, we have

$$\omega(\delta_u^{it}) = (\omega \star \omega)(\delta_u^{it}) = (\omega \otimes \omega)(\Delta_u(\delta_u^{it})) = (\omega(\delta_u^{it}))^2$$

(note that we need to use the strict extension of ω to $M(C_0^u(\mathbb{G}))$). Hence $\omega(\delta_u^{it})$ is either 0 or 1. However the map $t \mapsto \omega(\delta_u^{it})$ is continuous and $\omega(\delta_u^{i0}) = \omega(1) = 1$, so $\omega(\delta_u^{it}) = 1$ for every $t \in \mathbb{R}$.

Next we note that

$$\omega(\delta_u^{it}\delta_u^{-it}) = \omega(1) = 1 = \omega(\delta_u^{it})\omega(\delta_u^{-it}).$$

It follows that δ_u^{it} is in the multiplicative domain of ω for every $t \in \mathbb{R}$. Then $\delta_u^{it} \otimes \delta_u^{it}$ is in the multiplicative domain of $\omega \otimes \text{id} : M(C_0^u(\mathbb{G}) \otimes C_0^u(\mathbb{G})) \rightarrow M(C_0^u(\mathbb{G}))$. Therefore

$$\begin{aligned} L_\omega^u(\delta_u^{it}a) &= (\omega \otimes \text{id})(\Delta_u(\delta_u^{it}a)) = (\omega \otimes \text{id})((\delta_u^{it} \otimes \delta_u^{it})\Delta_u(a)) \\ &= ((\omega \otimes \text{id})(\delta_u^{it} \otimes \delta_u^{it}))((\omega \otimes \text{id})(\Delta_u(a))) = \delta_u^{it}L_\omega^u(a). \end{aligned}$$

□

Corollary 9. *Let $\omega \in \text{Prob}^u(\mathbb{G})$ be an idempotent state. Then for every $t \in \mathbb{R}$ and $a \in C_0(\mathbb{G})$*

$$L_\omega(\delta^{it}a) = \delta^{it}L_\omega(a) \quad \text{and} \quad L_\omega(a\delta^{it}) = L_\omega(a)\delta^{it}. \quad (5)$$

Proof. An immediate consequence of the previous lemma, the intertwining relation (3) and the fact that $\Lambda(\delta_u^{it}) = \delta^{it}$ for any $t \in \mathbb{R}$ (see [Kus2]). □

Lemma 10. *Let $\omega \in \text{Prob}^u(\mathbb{G})$ be an idempotent state. For every*

$$a \in D_\phi := \{a \in C_0(\mathbb{G}) \mid a\delta^{1/2} \text{ is bounded and the closure } \overline{a\delta^{1/2}} \in \mathcal{N}_\phi\}$$

we have

$$L_\omega((a\delta^{1/2})^*\overline{a\delta^{1/2}}) = \overline{\delta^{1/2}L_\omega(a^*a)\delta^{1/2}}. \quad (6)$$

Proof. Fix $a \in D_\phi$. As $a\delta^{1/2}$ is bounded, also $\delta^{1/2}a^*$ is bounded and equal to $(a\delta^{1/2})^*$ (by Corollary 8.35 of [Kus1]). Hence

$$\delta^{1/2}a^*a\delta^{1/2}$$

is bounded. By Proposition 9.24 of [StZ] (applied to $A = \delta$, $B = \delta^{-1}$), the map

$$it \mapsto \delta^{it}a^*a\delta^{it}$$

has wo-continuous extension to the strip $\mathcal{S} := \{z \in \mathbb{C} : 0 \leq \text{Re } z \leq 1/2\}$ that is analytic on the interior of \mathcal{S} . By Corollary 9,

$$L_\omega(\delta^{it}a^*a\delta^{it}) = \delta^{it}L_\omega(a^*a)\delta^{it}. \quad (7)$$

We intend to apply Proposition 9.24 of [StZ] again, this time to the function defined by the right-hand side of (7). We need to show that the map

$$it \mapsto \delta^{it}L_\omega(a^*a)\delta^{it}$$

has a wo-continuous extension to \mathcal{S} that is analytic on the interior. Indeed by (7) it is enough to show that L_ω is wo-continuous (analyticity is clear as L_ω is bounded). We may restrict our considerations to $L^\infty(\mathbb{G})$ because (the closure of) $\delta^z a^* a \delta^z$ is in $L^\infty(\mathbb{G})$ for $z \in \mathcal{S}$. As $L^\infty(\mathbb{G})$ is in the standard form on $L^2(\mathbb{G})$, a map on $L^\infty(\mathbb{G})$ is wo-continuous if and only if it is normal. But L_ω is normal by Theorem 6. Hence Proposition 9.24 of [StZ] is applicable and shows that the operator

$$\delta^{1/2}L_\omega(a^*a)\delta^{1/2}$$

is bounded (and densely defined as its domain is a core of $\delta^{1/2}$). The claimed equality follows from (7) and the uniqueness of analytic extensions. □

Proof of Theorem 2. Let $\omega \in \text{Prob}^u(\mathbb{G})$ be an idempotent state. By Proposition 7, L_ω preserves the left Haar weight. For the right Haar weight we use the fact that on the formal level we have the equality $\psi = \phi(\delta^{1/2} \cdot \delta^{1/2})$. Indeed, with Lemma 10 in hand, we can repeat the second part of the proof of Proposition 3.12 in [SaS] to conclude the argument. We outline the main steps for the convenience of the reader: first, for all $a \in D_\phi$ we have $\psi(a^*a) < \infty$ and by (6) also $\psi(L_\omega(a^*a)) < \infty$ (see Corollary 8.35 of [Kus₁]). Then applying ϕ to equality (6) gives $\psi(a^*a) = \psi(L_\omega(a^*a))$. Finally, due to the uniqueness of the right Haar weight and density of the elements of the form a^*a for $a \in D_\phi$ in $C_0(\mathbb{G})_+$, we deduce that $\psi = \psi \circ L_\omega$. \square

The main theorem is proved with the help of Theorem 2, but we will also need a few additional preliminary results.

Lemma 11. *Let $\mu \in \text{Prob}^u(\mathbb{G})$. Then μ is an idempotent state if and only if L_μ is a conditional expectation if and only if $L_\mu|_{C_0(\mathbb{G})}$ is a conditional expectation.*

Proof. If either L_μ or $L_\mu|_{C_0(\mathbb{G})}$ is a conditional expectation, then L_μ is an idempotent map (in the second case by weak*-continuity). Thus applying formula (4) and injectivity of the map $\mu \mapsto L_\mu$ shows that μ must be an idempotent state.

Assume then that $\mu \in \text{Prob}^u(\mathbb{G})$ is an idempotent state. We need to show that the images of the idempotent maps L_μ and $L_\mu|_{C_0(\mathbb{G})}$ are algebras (hence C^* -algebras). This is equivalent to showing that for any $a, b \in L^\infty(\mathbb{G})$

$$L_\mu(a)L_\mu(b) = L_\mu(L_\mu(a)b)$$

because a norm-one positive projection onto a C^* -subalgebra is a conditional expectation. Normality of L_μ and weak*-density of $C_0(\mathbb{G})$ in $L^\infty(\mathbb{G})$ mean that it suffices to establish the displayed formula for $a, b \in C_0(\mathbb{G})$. As the reducing morphism is a surjective *-homomorphism and we have the relation (3), it suffices to show

$$L_\mu^u(x)L_\mu^u(y) = L_\mu^u(L_\mu^u(x)y), \quad x, y \in C_0^u(\mathbb{G}).$$

For that however we can follow word by word the proofs in Lemma 2.3 and Theorem 2.4 of [SaS], as they use only the general properties of the coproducts and the quantum cancellation rules, which hold also for $C_0^u(\mathbb{G})$ (see [Kus₂]). \square

The following observation is due to Matthew Daws. Note that the positivity is only used to make sense of the intertwining relation (which otherwise could be formulated in the weak sense).

Lemma 12. *Let $Z : C_0(\mathbb{G}) \rightarrow C_0(\mathbb{G})$ be a completely positive nondegenerate map such that $\Delta \circ Z = (Z \otimes \text{id}) \circ \Delta$. Then there exists a (unital completely positive) left multiplier L of $L^\infty(\mathbb{G})$ such that $Z = L|_{C_0(\mathbb{G})}$.*

Proof. Recall that $L^1(\mathbb{G})$ is a closed, weak*-dense ideal in the Banach algebra $C_0(\mathbb{G})^*$. The linear span of the products of elements in $L^1(\mathbb{G})$ is dense in $L^1(\mathbb{G})$ (see for example [DaS], although the result dates back already to [KuV]). The assumed intertwining relation on Z implies that for any $\omega_1, \omega_2 \in C_0(\mathbb{G})^*$ we have $Z^*(\omega_1 \star \omega_2) = Z^*(\omega_1) \star \omega_2$. In particular for $\omega_1, \omega_2 \in L^1(\mathbb{G})$ we have $Z^*(\omega_1 \star \omega_2) = Z^*(\omega_1) \star \omega_2 \in L^1(\mathbb{G})$ by the ideal property. As Z^* is bounded, it means $Z^* : L^1(\mathbb{G}) \rightarrow L^1(\mathbb{G})$ and we can consider $L := (Z^*|_{L^1(\mathbb{G})})^*$. It is clear that $L : L^\infty(\mathbb{G}) \rightarrow L^\infty(\mathbb{G})$ is a normal map, and by weak*-density of $L^1(\mathbb{G})$ in $C_0(\mathbb{G})^*$ it follows that $L|_{C_0(\mathbb{G})} = Z$. This means in particular that L is unital and completely positive. Now

the fact that L is a left multiplier of $L^\infty(\mathbb{G})$ follows by condition (i) in Theorem 6, as Z^* was a left module map. \square

Proposition 13. *Let \mathcal{C} be a right-invariant ψ -expected C^* -subalgebra of $C_0(\mathbb{G})$. Then there exists an idempotent state $\omega \in \text{Prob}^u(\mathbb{G})$ such that $\mathcal{C} = L_\omega(C_0(\mathbb{G}))$.*

Proof. Let E be the ψ -preserving conditional expectation onto \mathcal{C} . Arguing as in Proposition 3.3 of [SaS], we obtain equality $ER_\nu = R_\nu E$ for all $\nu \in L_\#^1(\mathbb{G}) \subset C_0(\mathbb{G})^*$. Then the argument in the proof of the implication (iii) \implies (ii) of Lemma 1.6 in [SaS] yields the commutation relation (1) on $C_0(\mathbb{G})$. Lemma 12 implies that E is a restriction of a left multiplier of $L^\infty(\mathbb{G})$, i.e. a map of the form L_ω for some $\omega \in \text{Prob}^u(\mathbb{G})$. Finally, by Lemma 11, ω must be an idempotent state. \square

The next result is an analog of the last one in the context of von Neumann subalgebras of $L^\infty(\mathbb{G})$; it is much simpler.

Proposition 14. *Let \mathcal{D} be a right-invariant ψ -expected von Neumann subalgebra of $L^\infty(\mathbb{G})$. Then there exists an idempotent state $\omega \in \text{Prob}^u(\mathbb{G})$ such that $\mathcal{D} = L_\omega(L^\infty(\mathbb{G}))$.*

Proof. Let E be the ψ -preserving conditional expectation onto \mathcal{C} . Arguing as in Proposition 3.3 of [SaS], we obtain equality $ER_\nu = R_\nu E$ for all $\nu \in L_\#^1(\mathbb{G})$. Since $L_\#^1(\mathbb{G})$ is dense in $L^1(\mathbb{G})$, it follows that $E_* : L^1(\mathbb{G}) \rightarrow L^1(\mathbb{G})$ is a left module map. Hence, by Theorem 6, $E = L_\omega$ for some $\omega \in \text{Prob}^u(\mathbb{G})$. Once again, by Lemma 11, ω must be an idempotent state. \square

Proof of Theorem 1. Given an idempotent state $\omega \in \text{Prob}^u(\mathbb{G})$, we associate to it the algebras $\mathcal{C} := L_\omega(C_0(\mathbb{G}))$ and $\mathcal{D} := L_\omega(L^\infty(\mathbb{G}))$. They are respectively a nondegenerate C^* -subalgebra of $C_0(\mathbb{G})$ and a von Neumann subalgebra of $L^\infty(\mathbb{G})$. As left multipliers commute with right multipliers, both \mathcal{C} and \mathcal{D} are right invariant in the appropriate sense. By Lemma 11 the map L_ω is a normal conditional expectation onto \mathcal{D} and when restricted to $C_0(\mathbb{G})$ a conditional expectation onto \mathcal{C} . By Theorem 2 it preserves both the left and the right Haar weight. This means that to any idempotent state we can associate a right-invariant expected C^* -subalgebra of $C_0(\mathbb{G})$ and a right-invariant expected von Neumann subalgebra of $L^\infty(\mathbb{G})$.

Conversely, if we are given a right-invariant expected C^* -subalgebra \mathcal{C} of $C_0(\mathbb{G})$, Proposition 13 shows that the relevant conditional expectation is of the form L_ω for some idempotent state $\omega \in \text{Prob}^u(\mathbb{G})$. Proposition 14 gives a similar implication in the $L^\infty(\mathbb{G})$ -case.

The fact that these correspondences are bijective follows from the uniqueness of conditional expectations preserving faithful weights and the injectivity of the map $\mu \mapsto L_\mu$.

This establishes the bijections between objects in (i), (ii) and (iii). It is clear that working with right multipliers we could establish similarly bijections between objects in (i), (iv) and (v). \square

Proof of Corollary 3. Given a right-invariant ψ -expected C^* -subalgebra \mathcal{C} of $C_0(\mathbb{G})$, we see from Proposition 13 that it is of the form $L_\omega(C_0(\mathbb{G}))$ for some idempotent state $\omega \in \text{Prob}^u(\mathbb{G})$. Then Theorem 1 and its proof show that \mathcal{C} is indeed expected. The von Neumann algebra argument is identical, once we use Proposition 14. \square

Proof of Proposition 4. We begin with the scaling group. Proposition 9.2(2) of [Kus₂] shows that we have the following equality for any $t \in \mathbb{R}$:

$$\Delta_u \circ \sigma_t^u = (\tau_t^u \otimes \sigma_t^u) \circ \Delta_u.$$

This easily implies that for any $\mu \in M^u(\mathbb{G})$ and $t \in \mathbb{R}$ we have

$$L_{\mu \circ \tau_t^u}^u = \sigma_{-t}^u \circ L_\mu^u \circ \sigma_t^u.$$

Applying the reducing morphism and using the relation $\sigma_t \circ \Lambda = \Lambda \circ \sigma_t^u$, also shown in [Kus₂], yields

$$L_{\mu \circ \tau_t^u} = \sigma_{-t} \circ L_\mu \circ \sigma_t, \quad t \in \mathbb{R}. \quad (8)$$

For an idempotent state $\omega \in \text{Prob}^u(\mathbb{G})$, the conditional expectation L_ω preserves the left Haar weight, so by Takesaki's theorem L_ω commutes with the modular automorphisms σ_t . An application of the formula (8) yields the equality $L_{\omega \circ \tau_t^u} = L_\omega$, which by the injectivity of the map $\mu \mapsto L_\mu$ implies that $\omega = \omega \circ \tau_t^u$ for all $t \in \mathbb{R}$.

The proof that $\omega = \omega \circ S^u$ follows exactly as in Proposition 2.6 of [SaS], working all the time with the semi-universal unitary and using in the last step Corollary 9.2 of [Kus₂], which describes the core of S^u . Finally the fact that ω preserves the unitary antipode follows as the latter can be expressed as (extension of) the composition of S^u and $\tau_{i/2}^u$. \square

In the following proof, we will use certain results of [KaK]. Apparent differences in terminology stem from the different choice of conventions regarding multiplicative unitaries in that paper.

Proof of Theorem 5. (i) \implies (ii) This follows as in the proof of the analogous implication of Theorem 3.7 of [SaS], using the fact that the null space of the Haar state of a compact quantum group is a two-sided ideal, observed already in [Wor].

(ii) \implies (i) Again one can use the same ideas as in [SaS]. We sketch the outline of the argument: one considers the new C^* -algebra $\mathcal{B} := C_0^u(\mathbb{G})/N_\omega$ with the canonical quotient map $\pi : C_0^u(\mathbb{G}) \rightarrow \mathcal{B}$. Choosing an element $e \in C_0^u(\mathbb{G})_+$ such that $\omega(e) = 1$, we deduce first that e is in the multiplicative domain of the map L_ω^u (using the proof of Lemma 11) and then that $\pi(L_\omega(e))$ is a unit of \mathcal{B} . In the next step we establish the fact that there exists a faithful state $\mu \in \mathcal{B}^*$ such that $\omega = \mu \circ \pi$. This in turn implies that there is a well-defined map $\Delta_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$ such that

$$\Delta_{\mathcal{B}} \circ \pi = (\pi \otimes \pi) \circ \Delta.$$

Since the quantum cancellation properties hold on the $C_0^u(\mathbb{G})$ -level (see [Kus₂]), it follows that the pair $(\mathcal{B}, \Delta_{\mathcal{B}})$ satisfies the Woronowicz's axioms (with μ playing the role of the bi-invariant state). Therefore there exists a compact quantum group \mathbb{H} such that $\mathcal{B} = C(\mathbb{H})$. It is then fairly standard to check (see Proposition 3.5 in [Daw₃]) that the map π lifts to a quantum group morphism $\pi^u : C_0^u(\mathbb{G}) \rightarrow C^u(\mathbb{H})$ (which will still be surjective by Theorem 3.6 of [DKSS]). Hence $\omega = h_{\mathbb{H}} \circ \pi^u$ is a Haar idempotent.

(i) \implies (iii) If ω is a Haar idempotent, then the right-hand version of Theorem 5.14 of [KaK] shows that the algebra $\mathcal{C}^u := L_\omega^u(C_0^u(\mathbb{G}))$ satisfies the universal symmetry condition, i.e.

$$\mathcal{W}^*(\mathcal{C} \otimes 1)\mathcal{W} \subset M(\mathcal{C}^u \otimes C_0(\widehat{\mathbb{G}})). \quad (9)$$

Applying to the inclusion above the reducing morphism Λ tensored by the identity and noting that $L_\omega(C_0(\mathbb{G})) = \Lambda(L_\omega^u(C_0^u(\mathbb{G})))$ shows that $L_\omega(C_0(\mathbb{G}))$ is symmetric.

(iii) \implies (i) This is essentially contained in the proof of Theorem 5.15 in [KaK]: one first uses Proposition 3.13 of that paper to show that $\mathcal{C}^u := L_\omega^u(C_0^u(\mathbb{G}))$ satisfies the universal symmetry condition (9) and then applies Theorem 5.14 of [KaK] to deduce that ω is a Haar idempotent. \square

Finally note that (as mentioned in the proof) Theorem 5.14 of [KaK] provides a version of the correspondence (i) \iff (iii) valid on the universal level. The coamenable version of this correspondence was first noted in [Sal₂] (with compact quantum subgroups instead of Haar idempotents).

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